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Instability of Time Varying and Nonlinear Feedback Systems

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SUMMARY
Instability theorems are derived for a class of feedback systems, having a time varying or nonlinear feedback gain, in an otherwise linear system. The obtained criteria are related to well known stability counterparts, and a geometrical interpretation of the results is found in the complex plane.

1. Introduction

Instability theorems for time varying and nonlinear feedback systems have been reported [1], that provide counterparts of well known stability criteria, such as the circle criteria [2] and Popov's stability theorem [3]. These results are all stated in terms of a geometrical condition, imposed in the complex plane, upon the frequency response $H(j\omega)$ of the linear part of the system. In this paper a set of criteria is derived, that prove instability upon satisfaction of a similar geometrical condition, however imposed upon $H(j\omega+r)$, where r is a real parameter. For $r=0$, these theorems become identical to the known results. But many times, substantial improvements are obtained by choosing r different from zero. Furthermore, an optimal choice of r can often be determined on purely geometrical considerations in the complex plane.

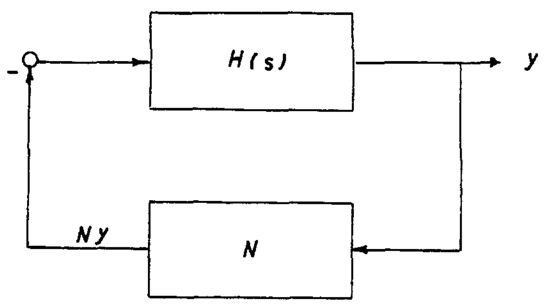

Fig. 1. The class of systems under investigation.

The block diagram of the system under investigation is shown in fig. 1. We shall assume that the transfer function $H(s)$ is a rational function:

$$H(s) = \frac{q(s)}{p(s)}, \tag{1}$$

where the polynomials $q(s)$ and $p(s)$ have no common factors, and degree $q(s) <$ degree $p(s)$. The operator N represents a time varying and/or nonlinear gain: $Ny=k(t, y)\cdot y$, or $Ny=f(y)$. We shall first deal with the case of time varying systems.

2. Time Varying Systems

If we let $y=q(D)z$, the system equation, of order n, takes the form:

$$p(D)z+N(q(D)z)=0 \qquad D = \frac{d}{dt} \tag{2}$$

or, for a time varying gain:

$$p(D)z+k(t, y)q(D)z=0\,. \tag{3}$$

In vector representation, (3) can be written as:

$$\dot{x}=Ax-k(t, y)bc'x \tag{4}$$

if the matrix A and the vectors b and c are chosen such that [4]:

$$c'(sI-A)^{-1}b=H(s)\,. \tag{5}$$

Let us consider the system (4), and let us try to prove that the equilibrium state at the origin is unstable. We shall proceed as follows:

Assume that, in state space, a closed, nonempty region $U_V(x)$ can be found, such that:

(i) $U_V(x)$ does not contain the origin $x=0$.

(ii) If $x_0 \in U_V(x)$, then $x(t, x_0, t_0) \in U_V(x)$ for all $t \geqq t_0$. $x(t, x_0, t_0)$ is the solution of (4) with initial conditions $x=x_0$ at $t=t_0$. Then it is clear that the origin is unstable, in the sense that the trajectory $x(t, x_0, t_0)$ remains outside a bounded neighbourhood of the origin, for all $t \geqq t_0$. Next assume that, in addition to these conditions, a scalar function $V(x)$ can be found, having the property that $|x| \to \infty$ if $|V(x)| \to \infty$, and such that:

(iii) either $\dot{V}(x) \geqq \varepsilon > 0$ for all $x \in U_V(x)$, or $\dot{V}(x) \leqq -\varepsilon < 0$ for all $x \in U_V(x)$ (6)

Then the origin is unstable in the sense that $|x(t, x_0, t_0)| \to \infty$ as $t \to +\infty$. $U_V(x)$ will be defined as: $U_V(x)=(x; V(x) \leqq -P<0)$*, where $V(x)=x'Vx$; $V'=V$, is a quadratic form in the system's state variables, that must be neither positive definite, nor positive semidefinite, since $U_V(x)$ must be nonempty. In order to satisfy (ii), we shall require that a scalar $r \geqq 0$ can be found, such that along the solutions of (4):

$$\dot{V}(x)-2rV(x) \leqq 0 \quad \text{for all } x \tag{7}$$

(7) implies that, if $V(x)=-P$, then $\dot{V}(x) \leqq -2rP \leqq 0$, which means that, at the boundary of $U_V(x)$, the vector field points to the inside of $U_V(x)$. Hence no trajectory, starting inside $U_V(x)$ at $t=t_0$, can leave $U_V(x)$ as time increases. Finally, if $r>0$, we have $\dot{V}(x) \leqq -2rP<0$ for $x \in U_V(x)$ such that (iii) is also satisfied.

Now let us find $\dot{V}(x)-2rV(x)$. Using (4) we have:

$$\begin{aligned}\dot{V}(x) &= x'V(Ax-k(t, y)bc'x)+(x'A'-k(t, y)b'c'x)Vx \\ &= x'(VA+A'V)x-2k(t, y)(Vb)'xc'x\end{aligned}$$

$$\begin{aligned}\dot{V}(x)-2rV(x) &= x'(VA+A'V)x-2k(t, y)(Vb)'xc'x-x'(V\cdot rI+rI\cdot V)x \\ &= x'(V(A-rI)+(A-rI)'V)x-2k(t, y)(Vb)'xc'x\end{aligned}$$

Hence $\dot{V}(x)-2rV(x)$ equals the derivative of $V(x)$ along the solutions of the equation

$$\dot{x}=A_r x-k(t, y)bc'x \qquad A_r=A-rI \tag{8}$$

(8) is equivalent with:

$$p(D+r)z+k(t, y)q(D+r)z=0 \tag{9}$$

So the instability of the origin is established if a quadratic form $x'Vx$ can be found, that is neither positive definite, nor positive semidefinite, and whose derivative along the solutions of (8) is smaller than or equal to zero. Conditions under which such a quadratic form exists are derived in Appendix 1, yielding the following

Circle Theorem 1.

Assume that

* read: $U_V(x)$ is the set of all points x, such that $V(x) \leqq -P$.

(1) the open loop system, with transfer function $H(s)$, is stable.
(2) the feedback gain $k(t, y)$ satisfies a limitation of the form

$$0 \leqq \alpha \leqq k(t, y) \leqq \beta ,$$

where α and β are scalar constants.
(3) a scalar $r \geqq 0$ can be found, such that the plot $H(j\omega + r)$ $(-\infty \leqq \omega \leqq +\infty)$ does not intersect the open disk $\Gamma(\alpha, \beta)$, drawn on the segment $(-1/\alpha, -1/\beta)$* and encircles it at least once in the clockwise direction.

Then the closed loop system is:
1. if $r=0$
unstable in the sense that there exists a set of initial states x_0, such that

$$|x(t, x_0, t_0)| \geqq \eta(x_0) > 0 \quad \text{for all} \quad t \geqq t_0 .$$

2. if $r>0$
unstable in the sense that there exists a set of initial states x_0, such that

$$|x(t, x_0, t_0)| \to \infty \quad \text{as} \quad t \to +\infty .$$

Since x_0 can be chosen arbitrarily close to the origin (by choosing $|P|$ sufficiently small) the theorem implies that the origin is
—not asymptotically stable in the sense of Liapunov if $r=0$
—not stable in the sense of Liapunov if $r \neq 0$.

3. Example

As an example, consider the system shown in fig. 2, with open loop transfer function

$$H(s) = \frac{(s-a)^2}{(s+b)^3} \qquad a, b > 0 .$$

For a linear, constant gain k, Nyquist's theorem predicts closed loop instability if either:

$$k > k_1$$

or

$$k < \max(k_0, k_2)$$

$$\text{with} \quad k_0 = -\frac{b^3}{a^2}; \quad \begin{matrix} k_1 \\ k_2 \end{matrix} = \frac{1}{4a}\left\{-d \pm (d^2 + 64ab^3)^{\frac{1}{2}}\right\}, \quad d = a^2 + 6ab - 3b^2 .$$

Now let us find bounds k_1' and k_2', such that for a time varying gain $k(t, y)$, instability is guaranteed if $k(t, y) \geqq k_1'$ or $k(t, y) \leqq k_2'$.

In fig. 2 the shape of $H(j\omega + r)$ $(0 \leqq \omega \leqq +\infty)$ is drawn for different values of r. One sees readily that if $k(t, y) > 0$, it is useless to take $r \neq 0$. The best result is obtained with $r=0$. The instability of the origin is proved for $k(t, y) \geqq k_1'$ if $H(j\omega)$ $(-\infty \leqq \omega \leqq +\infty)$ does not intersect the open disk $\Gamma_1(k_1', +\infty)$. A side calculation shows that this condition is satisfied if

$$b^3 + 3ba^2 + 6ab^2 - 2a^2 k_1' \leqq 2[(3b + 2a + k_1')(b^3 a^2 + a^4 k_1')]^{\frac{1}{2}} ; \qquad k_1' > k_1 .$$

Now let us turn to the case $k(t, y) < 0$. It is easy to see that theorem 1 remains valid if $0 \geqq \beta \geqq k(t, y) \geqq \alpha$. Hence, if we first let $r=0$, the closed loop system is unstable for $k(t, y) \leqq k_2' < 0$, if $H(j\omega)$ does not intersect the disk $\Gamma_2(k_2', -\infty)$. After some calculations one finds that this condition is satisfied if the following relations hold:

$$k_2' \leqq -\frac{b^3}{a^2} \quad \text{and} \quad k_2' \leqq -(3b + 2a) . \tag{10}$$

* $\Gamma(\alpha, \beta)$ is centered at the point $-\frac{1}{2}((1/\alpha) + 1/\beta)$ and has the radius $\frac{1}{2}((1/\alpha) - 1/\beta)$. $H(j\omega + r)$ may have points in common with the boundary of $\Gamma(\alpha, \beta)$, but it must encircle the point $-1/\alpha$.

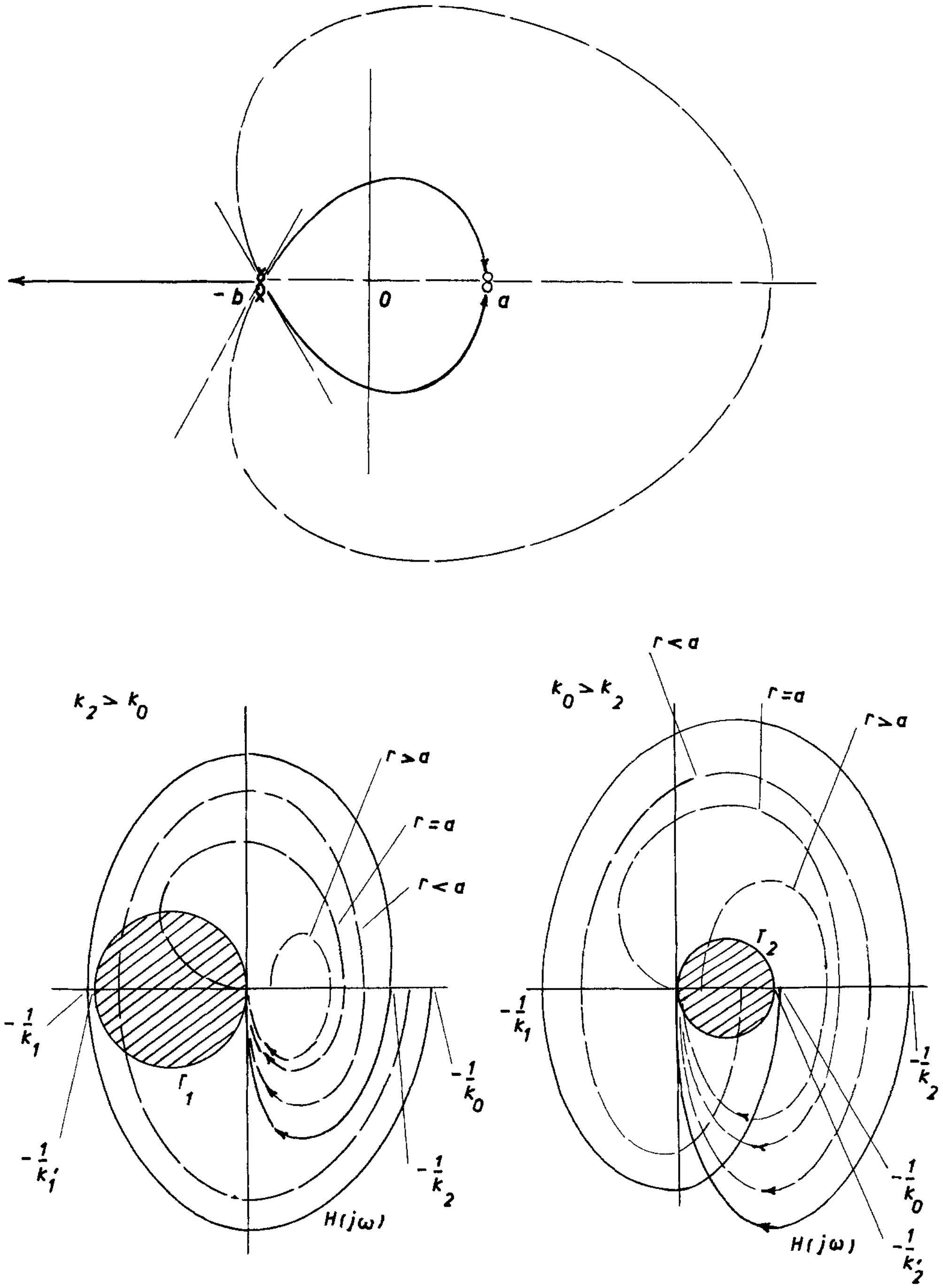

Fig. 2. Example 1: Root locus and frequency response of $H(j\omega)$. Shape of $H(j\omega+r)$.

Now let us choose $r\neq 0$. In fig. 2 one sees that the optimal choice of r equals $r=a$. Then requiring that $H(j\omega+a)$ does not intersect the disk $\Gamma_2(k_2', -\infty)$ yields the condition (fig. 3)

$$k_2' \leqq -3(a+b)\,. \tag{11}$$

Comparing (10) and (11) one sees that the results are improved by letting $r\neq 0$, if

$$3(a+b) < \frac{b^3}{a^2}\,.$$

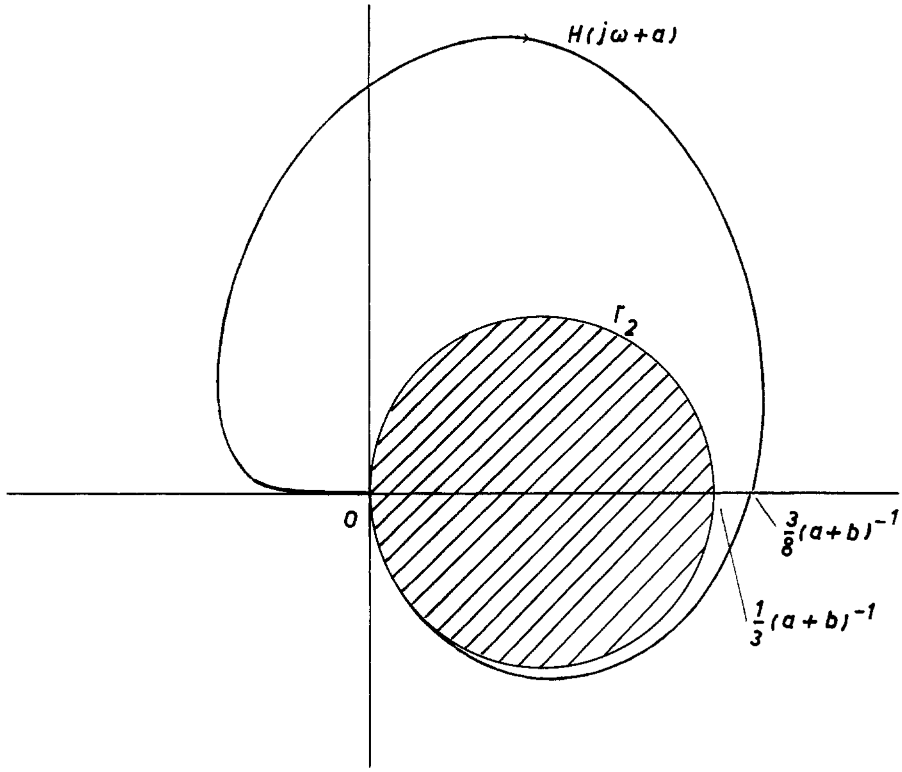

Fig. 3. Application of theorem 1.

4. Time Invariant Nonlinear Systems

We shall now specialise to the case, where the gain $k(t, y)$ is not explicitly dependent on time. The system equation (2) then takes the form:

$$p(D)z+f(q(D)z)=0\,, \tag{12}$$

where $f(y)=k(y)\cdot y$ is a time invariant, nonlinear function, with $f(0)=0$. Of course, theorem 1. applies to the equation (12). However we shall show that, for time invariant systems, the permissible range of the parameter r may be extended to negative values. Let

$$\dot{x}=Ax-bf(c'x) \tag{13}$$

be the vector representation of (12), and as before, let $V(x)=x'Vx$ be a quadratic form in the state variables, which is neither positive definite, nor positive semidefinite. As in par. 2 one finds:

$$\dot{V}(x)-2rV(x)=x'(V(A-rI)+(A-rI)'V)x-2(Vb)'xf(c'x)$$

which equals the derivative of $V(x)$ along the solutions of

$$\dot{x}=A_r x-bf(c'x) \qquad A_r=A-rI \tag{14}$$

which is equivalent with:

$$p(D+r)z+f(q(D+r)z)=0\,. \tag{15}$$

If we require that this derivative is smaller than or equal to zero, we have

$$\dot{V}(x)-2rV(x)\leqq 0\,. \tag{16}$$

For negative values of r, the only conclusion we can draw from (16) is that $\dot{V}(x)\leqq 0$ if $V(x)=0$, which implies that, if $U_V(x)=(x;\, V(x)\leqq 0)$, then no trajectory, starting inside $U_V(x)$ at $t=0$, can leave $U_V(x)$ as time increases. This result however is insufficient to prove instability: $U_V(x)$ contains the origin $x=0$, and it is still possible that all trajectories approach 0 as $t \to +\infty$. In order to prove instability, we must cut the origin out of $U_V(x)$. In other words, we wish to do the following:

1. define a closed set $U_C(x)$ which does not contain the origin.
2. make sure that $U_T(x) = U_V(x) \cap U_C(x)$ is nonempty, and that a trajectory, starting inside $U_T(x)$ remains in it as $t \to +\infty$.

We shall proceed as follows: We already know that, at the boundary of $U_V(x)$ the vector field points to the inside of $U_V(x)$. Now let $\partial U_C(x)$ be the boundary of $U_C(x)$, and let $\partial_T U_C(x)$ be the subset of $\partial U_C(x)$ which is part of the boundary of $U_T(x)$. In other words:

$$\partial_T U_C(x) = \partial U_C(x) \cap (x; V(x) \leqq 0) .$$

Now let us assume that, at all the points $x \in \partial_T U_C(x)$, the vector field points to the inside of $U_C(x)$. It is not hard to see that, under these conditions, the vector field points to the inside of $U_T(x)$, at all points x on the boundary of $U_T(x)$, which is the desired result. In Appendix 2, the conditions of existence of a set $U_C(x)$, with the desired properties are derived. It is shown that, upon the parameter r, a limitation is imposed of the form: $r > r_0$, where in general r_0 is a negative scalar. More specifically, the following result is obtained:

Circle Theorem 2.

Assume that

1. the open loop system, with transfer function $H(s)$, is stable.
2. the feedback function $f(y)$ lies completely within a sector defined by two straight lines with slopes $\alpha \geqq 0$ and $\beta > 0$:

$$0 \leqq \alpha y^2 \leqq y f(y) \leqq \beta y^2 .$$

3. the complete frequency response $H(j\omega)$ $(-\infty \leqq \omega \leqq +\infty)$ encircles the critical point $-1/K$ $m \geqq 1$ times in the clockwise direction, with $K = df(y)/dy|_{y=0}$.
4. a scalar $r < 0$ can be found, such that the plot $H(j\omega + r)$ does not intersect the open disk $\Gamma(\alpha, \beta)$, and encircles it $m - \lambda$ times in the clockwise direction*; λ is the number of poles of $H(s)$, whose real part lies in the interval $[0, r)$.

Then the equilibrium state at the origin is not stable in the sense that there exists a set of initial states x_0, such that $|x(t, x_0, 0) \geqq \eta > 0$ for all $t \geqq 0$.

x_0 can be chosen arbitrarily close to the origin. Hence the theorem implies that the origin is not asymptotically stable in the sense of Liapunov.

5. Example

Let us again consider the system of fig. 2, where we assume now that we have a time invariant nonlinear function in the feedback branch. Let us find conditions under which the closed loop system is unstable. First consider the case $yf(y) \geqq 0$ for all y. If we have $K = df(y)/dy|_{y=0} > k_1$ then $H(j\omega)$ encircles the critical point $-1/K$ twice in clockwise direction. Now it is easy to see that the best choice of r is $r = -b$. The plot of $H(j\omega - b)$ is shown in fig. 4. We have $\lambda = 0$, while $H(j\omega - b)$ encircles the open disk $\Gamma(\alpha, \beta)$ twice in clockwise direction, without intersecting it, for all pairs (α, β) for which $+\infty \geqq \beta \geqq \alpha \geqq 0$. Hence the closed loop system is unstable if

$$yf(y) \geqq 0 \text{ for all } y \qquad df(y)/dy|_{y=0} > k_1 .$$

Next assume that $yf(y) \leqq 0$ for all y. If $K < \min(k_0, k_2)$, then $H(j\omega)$ encircles the point $-1/K$ three times in the clockwise direction. Now let r be a negative number, with a large absolute value. The plot of $H(j\omega + r)$ is shown in fig. 4, for increasing values of $|r| > b$. We have $\lambda = 3$,

* with the restriction that, if $K = \alpha$ or $K = \beta$, $H(j\omega + r)$ must not be allowed to pass through the point $-1/K$.

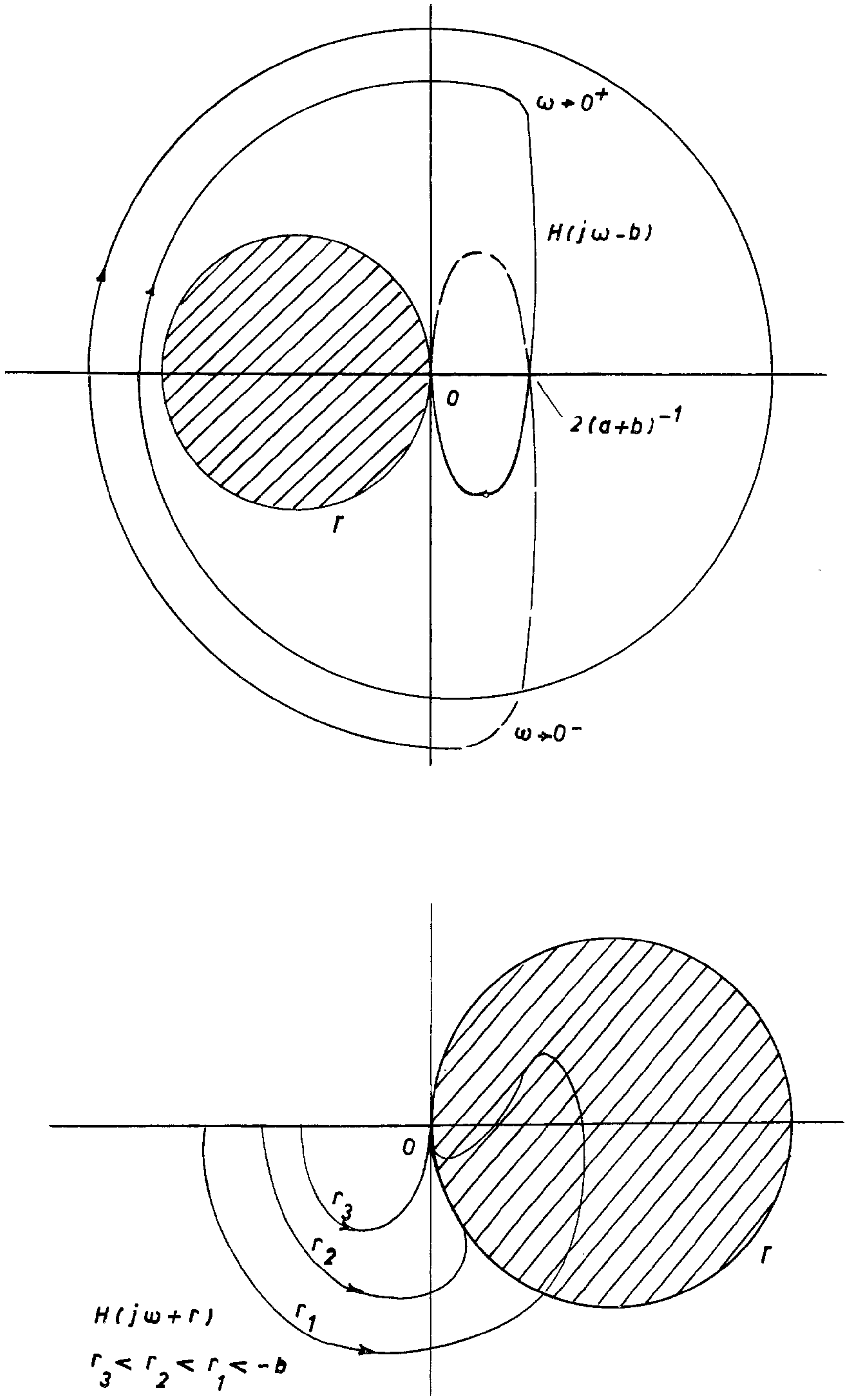

Fig. 4. Application of theorem 2.

while, for $|r|$ sufficiently large, the open disk $\Gamma(\alpha, \beta)$ is neither encircled, nor intersected by $H(j\omega+r)$, for any pair (α, β) for which $0 \geqq \beta \geqq \alpha \geqq -\infty$. So the closed loop system is also unstable if $yf(y) \leqq 0$ for all y $\quad df(y)/dy|_{y=0} < \min(k_0, k_2)$.

6. An Instability Counterpart of Popov's Throrem

The theorems derived in the preceding paragraphs, are instability counterparts of the circle criteria in stability theory. By a similar procedure, a corresponding instability result can be found for many other stability theorems. An important example consists in finding an instability counterpart of Popov's theorem. As is well known, Popov's theorem applies to systems having a time invariant nonlinear feedback $f(y)$, and relies on the use of a Liapunov function of the type

$$V(x)=x'Vx+\alpha\int_0^{c'x} f(\theta)\,d\theta\,, \tag{17}$$

where $x'Vx$ is a quadratic form and α a scalar constant. Hence we shall use a function $V(x)$ of the same type. Deriving $V(x)$ along the solutions of (13), and subtracting $2rV(x)$ yields:

$$\begin{aligned}\dot V(x)-2rV(x)&=x'(V(A-rI)+(A-rI)'V)x-2(Vb)'xf(c'x)\\&\quad+\alpha f(c'x)c'(Ax-bf(c'x))-2r\alpha\int_0^{c'x} f(\theta)\,d\theta\,.\end{aligned}$$

Now let us assume e.g. that the following relation holds:

$$0\leqq\int_0^y f(\theta)\,d\theta\leqq\tfrac{1}{2}yf(y)\quad\text{for all}\quad y\,. \tag{18}$$

Then we have:

$$\begin{aligned}\dot V(x)-2rV(x)&\leqq x'(V(A-rI)+(A-rI)'V)x-2(Vb)'xf(c'x)\\&\quad+\alpha f(c'x)c'((A-rI)x-bf(c'x))\quad\text{if}\quad r\alpha\leqq 0\,.\end{aligned} \tag{19}$$

Furthermore, the right hand side of (19) equals the derivative of $V(x)$ along the solutions of (14). First consider the case $r\geqq 0$; $\alpha<0$. Then we shall require that there exists a function $V(x)$ of the form (17), whose derivative along the solutions of (14) is smaller than or equal to zero, and such that the set $U_V(x)=(x;\,V(x)\leqq -P)$ is nonempty for some $P>0$. We then have: $\dot V(x)\leqq -2rP\leqq 0$ at the boundary of $U_V(x)$, which is the desired condition. If $r<0$; $\alpha>0$, we proceed in a similar way as in par. 4, defining $U_V(x)=(x;\,V(x)\leqq 0)$. We now have the additional problem of cutting the origin out of $U_V(x)$ by means of a suitable set $U_C(x)$. A detailed examination of the different conditions is carried out in Appendix 3, leading to the following result:

Theorem 3.

Assume that
1. the open loop system, with transfer function $H(s)$ is stable.
2. the feedback function $f(y)$ lies completely within a sector, defined by two straight lines with slopes $k_0=df(y)/dy|_{y=0}$ and k_1.
3. either $\int_0^y f(\theta)\,d\theta\geqq\frac{1}{2}yf(y)$, or $\int_0^y f(\theta)\,d\theta\leqq\frac{1}{2}yf(y)$ for all y★
4. two scalars r and α can be found, such that $r\alpha\leqq 0$, and such that

$$\mathrm{Re}(1+\alpha j\omega)\frac{1+k_1H(j\omega+r)}{1+k_0H(j\omega+r)}\geqq 0\quad\text{for all real }\omega \tag{20}$$

and such that
a. if $r\geqq 0$
$H(j\omega+r)\;(-\infty\leqq\omega\leqq+\infty)$ encircles the critical point $-1/k_0$ at least once in the clockwise direction.

★ This condition means that the surface between the curve $f(y)$ and the y-axis, is either not smaller than, or not larger than the triangle OYZ, for every point Z on $f(y)$ (fig. 5).

b. if $r<0$

$H(j\omega)$ $(-\infty \leqq \omega \leqq +\infty)$ encircles $-1/k_0$ $m \geqq 1$ times in the clockwise direction, and $H(j\omega+r)$ $(-\infty \leqq \omega \leqq +\infty)$ $m-\lambda$ times. Again λ is the number of poles of $H(s)$ with real part in the interval $(r, 0]$.

Then the equilibrium state at the origin is not asymptotically stable in the sense of Liapunov, if $r \leqq 0$ and not stable in the sense of Liapunov if $r>0$.

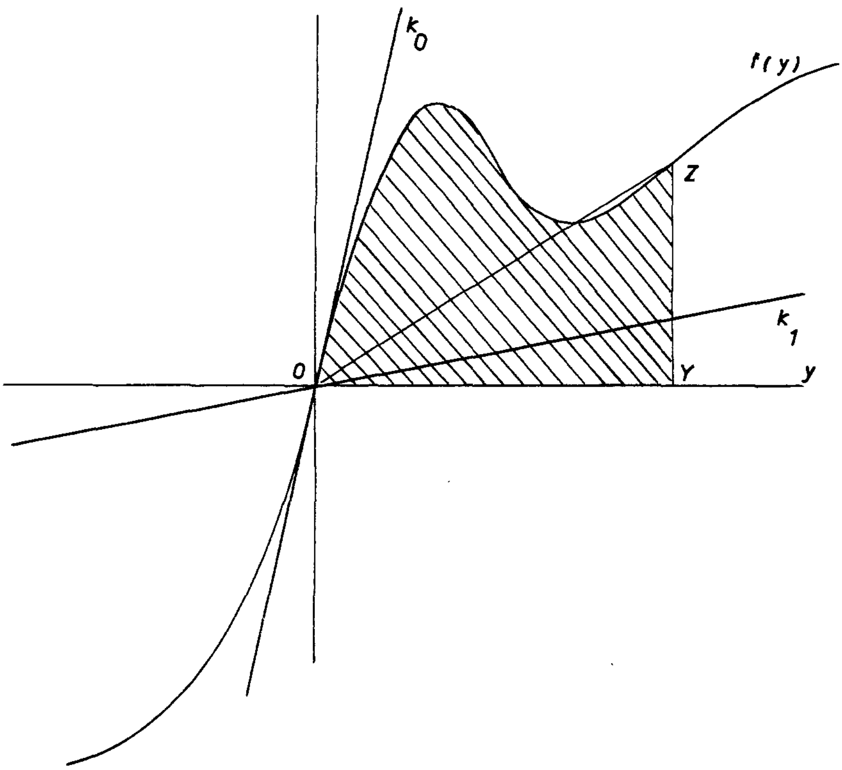

Fig. 5. A function $f(y)$ satisfying the conditions of theorem 3.

The geometrical interpretation of the condition (20) is well known: Let

$$H_l(s) = \frac{H(s)}{1+k_0 H(s)}$$

be the transfer function of the system, when linearized at the origin, and suppose e.g. that $k_1 > k_0$. Then (20) can be written as:

$$\frac{1}{k_1-k_0} + \mathrm{Re}(1+\alpha j\omega) H_l(j\omega+r) \geqq 0\,,$$

which means that it must be possible, in the complex plane, to draw a straight line with positive slope (if $r \leqq 0$), or negative slope (if $r \geqq 0$) through the point $-1/(k_1-k_0)$, such that the modified polar plot ω Im $H_l(j\omega+r)$ versus Re $H_l(j\omega+r)$ lies completely to the right of it.

7. Example

Let $(H(s); f(y))$ be a system of the considered type, with

$$H(s) = \frac{1+s+as^2}{s^3}$$

and $\quad f(y) = k_0 y + ky^3 \qquad a, k_0, k > 0\,.$

Let us look for the instability conditions. One finds easily that the complete frequency response $H(j\omega)$ $(-\infty \leqq \omega \leqq +\infty)$ encircles $-1/k_0$ twice in the clockwise direction if $k_0 < 1/a$.

Then

$$H_l(s) = \frac{q_l(s)}{p_l(s)} = \frac{1+s+as^2}{s^3+k_0(1+s+as^2)}$$

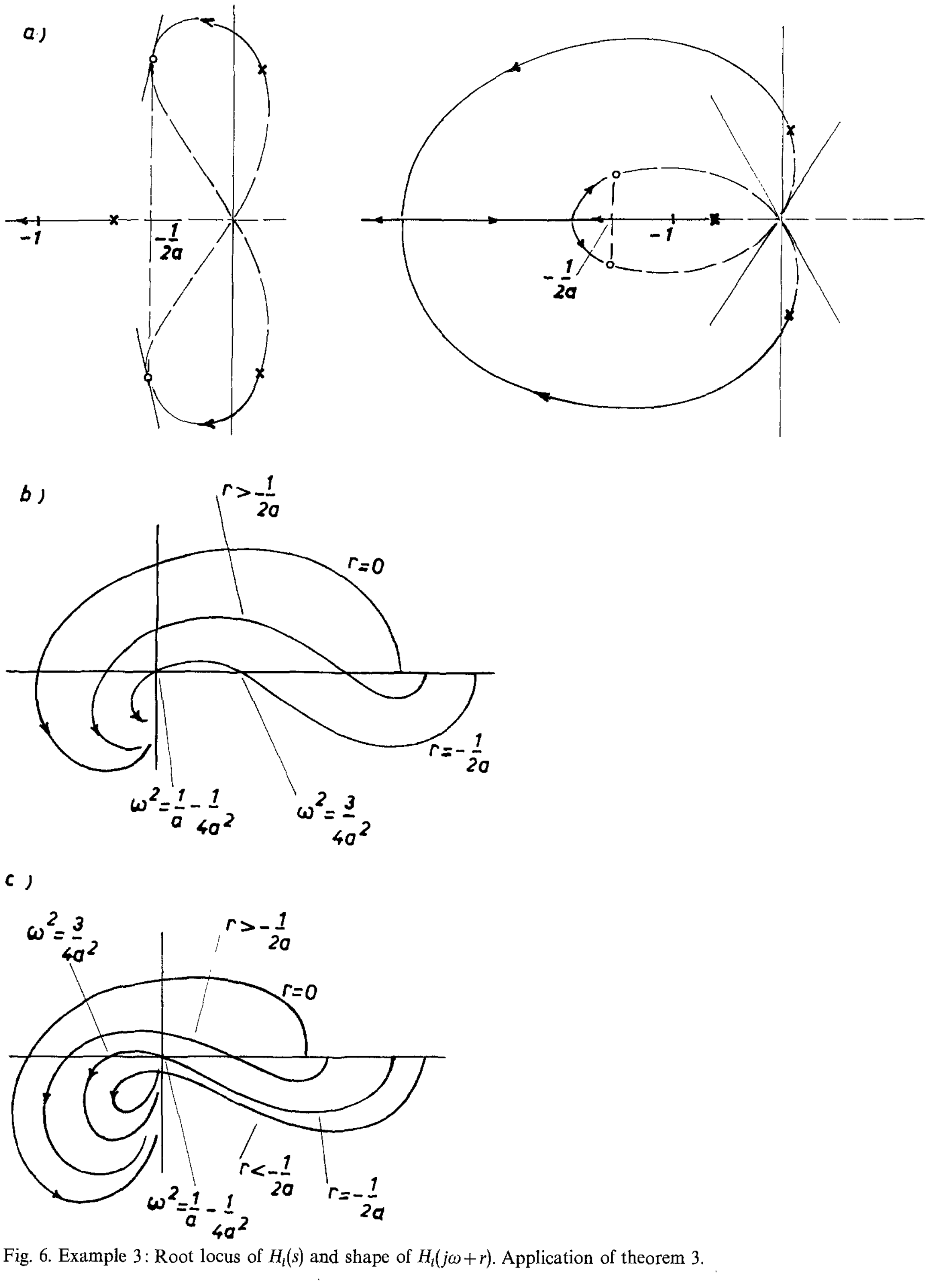

Fig. 6. Example 3: Root locus of $H_l(s)$ and shape of $H_l(j\omega+r)$. Application of theorem 3.

has two poles in the right half plane, and one real negative pole $-a_1$ in the left half plane. We shall choose $r<0$. Then the conditions of theorem 3, which require that $H_l(s)$ has no poles in the interval $[r, 0]$, limit the choice of r to the interval $0>r>-a_1$.

In fig. 6 the root locus of $H_l(s)$ and the shape of $H_l(j\omega+r)$ are drawn for different values of $r<0$. The condition to be satisfied is that

$$\text{Re } H_l(j\omega+r)-\alpha\omega \text{ Im } H_l(j\omega+r)\geqq 0 \quad \text{for all real } \omega \text{ and for some } \alpha>0\,. \tag{21}$$

First consider the case $a>1$ (fig. 6b). Clearly (21) cannot be satisfied unless

$$\text{Re } H_l(j\omega+r)\geqq 0 \quad \text{if} \quad \text{Im } H_l(j\omega+r)\geqq 0 \qquad (0\leqq\omega\leqq+\infty)$$

Hence we must have $r\leqq -1/2a$. This condition can be satisfied with $0>r>-a_1$ if $p_l(-1/2a)>0$, or:

$$k_0>\frac{1}{2a^2(4a-1)}\,.$$

Now it is easy to verify that under the latter condition (21) can be satisfied with $r=-\frac{1}{2}a$. After some calculations one finds

$$\text{Re } H_l\left(j\omega-\frac{1}{2a}\right)-\alpha\omega \text{ Im } H_l\left(j\omega-\frac{1}{2a}\right)\geqq 0\,,$$

$$\text{if} \quad k_0 \;\; 1-\frac{1}{4a}-\left(a\omega^2\right)^2+\left(1-\frac{1}{4a}-a\omega^2\right)\left[\alpha\omega^2\left(\frac{3}{4a^2}-\omega^2\right)-\left(\frac{1}{8a^3}-\frac{3}{2a}\omega^2\right)\right]\geqq 0\,,$$

which is true for

$$\alpha=\left[\frac{\dfrac{1}{8a^3}-\dfrac{3}{2a}\omega^2}{\omega^2\left(\dfrac{3}{4a^2}-\omega^2\right)}\right]_{\omega^2=(1-\frac{1}{4}a^{-1})/a}>0\,.$$

Next let us turn to the case $a\leqq 1$ (fig. 6c). Now we must require that r can be chosen such that $\text{Im}H_l(j\omega+r)<0$ for all $\omega>0$. A straightforward investigation shows that under this condition (21) can be satisfied by choosing α large enough. Furthermore we find Im $H_l(j\omega+r)<0$ for all $\omega>0$, if

$$(1-2r-2ar^2)^2<4ar^2(3+2r+ar^2) \quad \text{or} \quad -2r(2a^{\frac{1}{2}}-1)>1\,.$$

This requires $a>\frac{1}{4}$ and $p_l(-1/(4a^{\frac{1}{2}}-2))>0$, which yields:

$$k_0>\frac{1}{(4a^{\frac{1}{2}}-2)[(4a^{\frac{1}{2}}-2)^2-(4a^{\frac{1}{2}}-2)+a]}\,.$$

We conclude that the system is unstable if $k_0<1/a$ and if either $a>1$

$$k_0>\frac{1}{2a^2(4a-1)}$$

or $\frac{1}{4}<a\leqq 1$

$$k_0>\frac{1}{(4a^{\frac{1}{2}}-2)[(4a^{\frac{1}{2}}-2)^2-(4a^{\frac{1}{2}}-2)+a]}\,.$$

8. Conclusion

Instability criteria for time varying and nonlinear feedback systems have been determined, stated in terms of the plot $H(j\omega+r)$, where r is a parameter, to be chosen in some interval dependent on the pole—zero configuration of $H(s)$. For a suitable choice of r substantially better results may be obtained than those from existing criteria. A disadvantage is that the analytical expression of $H(s)$ must be available.

9. Appendix

1. *Proof of theorem* 1.

First consider the case $0 \leqq k(t, y) \leqq \beta$. Let $x = Mx_s$ be the linear transformation, that transforms the state vector x of equation (8) into the standard controllable state vector $x_s = |z\ Dz\ \dots\ D^{n-1}z|'$ of the equivalent equation (9), and assume that, by this transformation $V(x)$ is transformed into $V_s(x_s)$. Let

$$V_s(x_s) = \int_{t_1(0)}^{t_2(x_s)} [p_r(D)z(p_r(D)z + \beta q_r(D)z) - r^2(D)z]\,dt\,, \tag{22}$$

where $p_r(s) = p(s+r)$; $q_r(s) = q(s+r)$ and $r(s)$ is the spectral factor in the right half plane of the even part of $(p_r(s) + \beta q_r(s))p_r(-s)$.

It has been established [5] that $r(s)$ and $V_s(x_s)$ are defined, and that $V_s(x_s)$ is a quadratic form in the elements of x_s, if

$$\operatorname{Re}\left[1 + \beta\frac{q_r(j\omega)}{p_r(j\omega)}\right] = \operatorname{Re}[1 + \beta H(j\omega + r)] \geqq 0 \quad \text{for all real } \omega\,. \tag{23}$$

Furthermore,

> $V_s(x_s)$ is not positive definite or positive semidefinite if $1 + \beta H(j\omega + r)$ is not positive real, which is satisfied if $p(s+r)$ has at least one zero in the right half plane. (24)*

Deriving $V_s(x_s)$ along the solutions of (9) one finds:

$$\frac{dV_s(x_s)}{dt} = -k(t, y)(\beta - k(t, y))q_r^2(D)z - r^2(D)z \leqq 0\,.$$

Now assume $\alpha \leqq k(t, y) \leqq \beta$. Then we replace the system $(H(s); k(t, y))$ by the equivalent system $(H_0(s); k(t, y) - \alpha)$ where

$$H_0(s) = \frac{H(s)}{1 + \alpha H(s)} \quad \text{and} \quad 0 \leqq k(t, y) - \alpha \leqq \beta - \alpha\,.$$

If we apply the conditions (23) and (24) to this system, we find:

$$\operatorname{Re}\left[\frac{1 + \beta H(j\omega + r)}{1 + \alpha H(j\omega + r)}\right] \geqq 0 \quad \text{for all real } \omega \tag{25}$$

and

$$H_0(s+r) \text{ has at least one pole in the right half plane} \tag{26}$$

Under the assumptions of the theorem these analytical conditions translate into the indicated geometrical criteria.

2. *Proof of theorem* 2.

Again let us first consider the case $0 \leqq yf(y) \leqq \beta y^2$. As before, $V_s(x_s)$ is defined by (22), which requires that (23) and (24) are satisfied. Now let us rewrite the equation (13) in the form:

$$\dot{x} = A_l x - bg(c'x)\,, \tag{27}$$

* By "right half plane" we mean the open set Re $s > 0$.

where $A_l = A - Kbc'$
$g(y) = f(y) - Ky$
$K = df(y)/dy|_{y=0}$.

In view of (5), it is clear that A_l, b and c must be chosen such that

$$c'(sI - A_l)^{-1} b = H_l(s) = \frac{H(s)}{1 + KH(s)} .$$

Hence if $H_l(s)$ has $m \geqq 1$ poles in the right half plane, A_l may be defined as follows:

$$A_l = \begin{vmatrix} A_1 & 0 \\ 0 & A_2 \end{vmatrix}$$

with

$$A_1 = \begin{vmatrix} \rho_1 & \omega_1 & & & & & & & & & \\ -\omega_1 & \rho_1 & & & & & & & & & \\ & & \rho_2 & \omega_2 & & & & & & & \\ & & -\omega_2 & \rho_2 & & & & & & & \\ & & & & \ddots & & & & & & \\ & & & & & \rho_u & \omega_u & & & & \\ & & & & & -\omega_u & \rho_u & & & & \\ & & & & & & & \rho_{u+1} & & & \\ & & & & & & & & \ddots & & \\ & & & & & & & & & & \rho_{u+v} \end{vmatrix}$$

where $\rho_i \pm j\omega_i$ ($i = 1, \ldots, u$) and ρ_i ($i = u+1, \ldots, u+v$); $2u + v = m$ are the poles of $H_l(s)$ in the right half plane, and A_2 is defined similarly as A_1, for the poles whose real part is not positive★
Next the set $U_C(x)$ is defined as $U_C(x) = (x; C(x) \geqq 0)$, with

$$C(x) = x_1^2 + x_2^2 + \ldots + x_m^2 - \varepsilon^2 .$$

We shall suppose that

$$V(x)_{x_1 = x_2 = \ldots = x_m = 0} \text{ is positive definite in } x_{m+1}, \ldots, x_n \tag{28}$$

and, using this assumption, we shall prove the following statements:

1. The set $\partial_T U_C(x) = (x; C(x) = 0, V(x) \leqq 0)$ is confined to an arbitrarily small neighbourhood of the origin, if $|\varepsilon|$ is sufficiently small.
2. If $x \in \partial_T U_C(x)$, then $\dot{C}(x) \geqq 0$ along the solutions of (27).

After consecutively splitting off $x_n, x_{n-1}, \ldots, x_{m+1}$, one can easily verify that, under the assumption (28), $V(x)$ can be written as:

$$V(x) = Q(x_1, \ldots, x_m) + \sum_{i=1}^{n-m} z_i^2 ,$$

where z_1 ($i = 1, \ldots, n-m$) are linear combinations of $x_1, \ldots, x_n$ and $z_{i\,x_1 = \ldots = x_m = 0}$ are linearly independent combinations of $x_{m+1}, \ldots, x_n$.

$C(x) = 0$ implies $|x_i| \leqq \varepsilon$; $i = 1, \ldots, m$. If in addition $V(x) \leqq 0$, then

★ It is assumed that $H_l(s)$ has no multiple poles.

$$\sum_{i=1}^{n-m} z_i^2 \leqq -Q(x_1, \ldots, x_n) \leqq \eta^2(\varepsilon)$$

where $\eta^2(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Hence $|z_i| \leqq \eta(\varepsilon)$; $i=1, \ldots, n-m$. But this implies $|x_i| \leqq \xi(\varepsilon)$ with $\xi(\varepsilon) \to 0$ as $\varepsilon \to 0$; $i=1, \ldots, n$.

Now let us find $\dot{C}(x)$. We have:

$$\dot{C}(x) = 2\rho_1(x_1^2+x_2^2)+\ldots+2\rho_u(x_{2u-1}^2+x_{2u}^2)+2\rho_{u+1}x_{2u+1}^2+\ldots+2\rho_{u+v}x_m^2+2\sum_{i=1}^{m} b_i x_i g(c'x)\,. \tag{29}$$

If $x \in \partial_T U_C(x)$ and $\varepsilon \to 0$, then $|x| \to 0$. Hence the last term in the right hand side of (29) is infinitely small of the third order (since $g(y)/y \to 0$ as $y \to 0$), while the other terms form a positive sum that is infinitely small of the second order. It follows that we have: $\dot{C}(x) > 0$ for $x \in \partial_T U_C(x)$ and ε^2 sufficiently small.

So it remains to show that $V(x)_{x_1=\ldots=x_m=0}$ is positive definite in $x_{m+1}, \ldots, x_n$. Deriving $V(x) = x'Vx$ along the solutions of

$$\dot{x} = A_{lr}x - bg(c'x) \qquad A_{lr} = A_l - rI$$

which is equivalent with (14), yields:

$$\frac{dV(x)}{dt} = x'(VA_{lr}+A_{lr}'V)x - 2(Vb)'xg(c'x)\,. \tag{30}$$

On the other hand, the derivative of $V_s(x_s)$ along the solutions of (15) yields:

$$\frac{dV_s(x_s)}{dt} = p_r(D)z(p_r(D)z+\beta q_r(D)z) - r^2(D)z \tag{31}$$

with $p_r(D)z = -f(q_r(D)z)$

In (31) we have:

$$r(s)r(-s) = p_r(s)p_r(-s) + Ev\beta q_r(s)\,p_r(-s)\,.$$

Since degree $q_r(s) <$ degree $p_r(s)$, this implies that $r(s)$ can be written as:

$$r(s) = p_r(s) + r_1(s)\,, \tag{32}$$

where degree $r_1(s) \leqq n-1$. Substitution in (31) yields:

$$\begin{aligned}\frac{dV_s(x_s)}{dt} &= -r_1^2(D)z - p_r(D)z(\beta q_r(D)z - 2r_1(D)z)\\ &= -r_1^2(D)z - (\beta q_r(D)z - 2r_1(D)z)f(q_r(D)z)\\ &= -(r_1(D)z - Kq_r(D)z)^2 - K(\beta-K)q_r^2(D)z - g(q_r(D)z)(\beta q_r(D)z - 2r_1(D)z)\end{aligned} \tag{33}$$

since $f(y) = Ky + g(y)$. Comparing (30) and (33) yields the identity:

$$\begin{aligned}x'(VA_{lr}+A_{lr}'V)x &= -[(r_1(D)z - Kq_r(D)z)^2 + K(\beta-K)q_r^2(D)z]_{x_s=M^{-1}x}\\ &= -x'(vv'+ww')x \quad \text{since} \quad \beta \geqq K\,.\end{aligned} \tag{34}$$

If in (34) we let $x_1 =, \ldots, = x_m = 0$, we get:

$$x^{*\prime}(V_{22}(A_2-rI)+(A_2-rI)'V_{22})x^* = -x^{*\prime}(v^*v^{*\prime}+w^*w^{*\prime})x^*\,, \tag{35}$$

where the asterisk denotes a vector of dimension $n-m$, and where V_{22} is an $(n-m)$ square symmetric matrix, defined by

$$V = \begin{vmatrix} V_{11} & V_{12} \\ V_{12}' & V_{22} \end{vmatrix}\,.$$

We must show that V_{22} is positive definite. But the equation

$$V_{22}(A_2-rI)+(A_2-rI)'V_{22} = -(v^*v^{*\prime}+w^*w^{*\prime})$$

permits a unique positive definite solution $V_{22}=V_{22}'$ if A_2-rI has all its characteristic values in the left half plane, and if either the system

$$S_1:\quad \dot{x}=(A_l-rI)x \qquad y=v'x\,, \quad \text{or}$$
$$S_2^*:\quad \dot{x}^*=(A_2-rI)x^* \qquad y=w^{*\prime}x^*$$

is completely observable [4] [6]. This condition is satisfied a fortiori, if either

$$S_1:\quad \dot{x}=(A_l-rI)x \qquad y=v'x\,, \quad \text{or}$$
$$S_2:\quad \dot{x}=(A_l-rI)x \qquad y=w'x$$

is completely observable. But the systems S_1 and S_2 are equivalent with:

$$S_1:\quad p_r(D)z+Kq_r(D)z=0 \qquad y=r_1(D)z-Kq_r(D)z$$
$$S_2:\quad p_r(D)z+Kq_r(D)z=0 \qquad y=(K(\beta-K))^{\frac{1}{2}}q_r(D)z$$

S_1 is completely observable if $p_r(s)+Kq_r(s)$ and $r_1(s)-Kq_r(s)$, or, because of (32), if $p_r(s)+Kq_r(s)$ and $r(s)$ have no common factors. Since

$$2r(s)r(-s)=(p_r(s)+\beta q_r(s))p_r(-s)+(p_r(-s)+\beta q_r(-s))p_r(s)$$

this condition is clearly satisfied, if $K=\beta$ (Remember that $p(s)$ and $q(s)$ have no common factors). If $K\neq\beta$, then S_2 is completely observable since $q_r(s)$ and $p_r(s)+Kq_r(s)$ have no common factors.

So the only remaining condition is that A_2-rI has all its characteristic values in the left half plane. This means that $H_l(s)$ must have no poles whose real parts lie in the interval $[-r, 0]$.

If $\alpha y^2 \leqq yf(y) \leqq \beta y^2$, the conditions (23) and (24) are transformed again into the conditions (25) and (26). The only additional condition is that $H_l(s)$ and $H_l(s+r)$ both have $m \geqq 1$ poles in the right half plane, while all other poles of $H_l(s+r)$ are in the left half plane. This completes the proof.

3. *Proof of theorem* 3.

We first handle the case where $f(y)$ satisfies the conditions of theorem 3, with $k_0=df(y)/dy|_{y=0}=0$, and $0\leqq yf(y)\leqq k_1y^2$. Assume $r\geqq 0$, $\alpha<0$. We define:

$$V_s(x_s)=\int_{t_1(0)}^{t_2(x_s)}[(k_1(1+\alpha D)q_r(D)+p_r(D))zp_r(D)z-r^2(D)z]\,dt+\alpha\int_0^{q_r(D)z}f(\theta)d\theta\,, \tag{36}$$

where $r(s)$ is the spectral factor in the right half plane of the even part of $(k_1(1+\alpha s)q_r(s)+p_r(s))p_r(-s)$. This requires that

$$\mathrm{Re}\,(1+\alpha j\omega)(1+k_1H(j\omega+r))\geqq 0 \quad \text{for all real } \omega \tag{37}$$

while

the condition that $U_V(x)=(x;\ V(x)\leqq -P)$ is nonempty, requires that $H(s+r)$ has at least one pole in the right half plane. (38)

The derivative of $V_s(x_s)$ along the solutions of (15) is:

$$\frac{dV_s(x_s)}{dt} = -[k_1(1+\alpha D)q_r(D)z-f(q_r(D)z)]f(q_r(D)z)-r^2(D)z+\alpha f(q_r(D)z)Dq_r(D)z$$
$$= -(k_1q_r(D)z-f(q_r(D)z))f(q_r(D)z)-r^2(D)z\leqq 0\,.$$

Next we proceed with the case $r<0$; $\alpha>0$. Again $V_s(x_s)$ is defined by (36), leading to the condition (37). Furthermore note that, even if $\alpha>0$, the set $U_V(x)=(x;\ V(x)\leqq 0)$ is nonempty if (38) is satisfied.

Indeed, under this condition, x can be chosen such that $x'Vx<0$. Then if we let $|x|\to 0$, $|x'Vx|$ is infinitely small of the second order, while

$$\alpha\int_0^{q_r(D)z} f(\theta)d\theta \text{ is infinitely small of the third order, since } df(y)/dy|_{y=0}=0\,.$$

Hence if $|x|$ is small enough, the condition $V(x)\leqq 0$ can be satisfied. Now $U_C(x)$ is defined, exactly as in Appendix 2.* The set $\partial_T U_C(x)$ is still confined to an arbitrarily small neighbourhood of the origin, since now $U_V(x)$ is only a subset of the set $(x;\,x'Vx\leqq 0)$, and for $x\in\partial_T U_C(x)$ we still have $\dot{C}(x)>0$ for the same reason as before. So there only remains the condition that $x'Vx_{x_1=\ldots=x_m=0}$ is positive definite in $x_{m+1}\ldots x_n$. Reasoning along the same lines as in Appendix 2, one finds, instead of (30):

$$\frac{d}{dt}x'Vx=x'(VA_r+A_r'V)x-2(Vb)'xf(c'x) \tag{38}$$

and instead of (33)

$$\frac{d}{dt}x_s'V_sx_s=(k_1(1+\alpha D)q_r(D)z+p_r(D)z)p_r(D)z-r^2(D)z\,, \tag{39}$$

with $p_r(D)z=-f(q_r(D)z)$. Using the relation:

$$2r(s)r(-s)=2p_r(s)p_r(-s)+p_r(s)\cdot k_1(1-\alpha s)q_r(-s)+p_r(-s)\cdot k_1(1+\alpha s)q_r(s) \tag{40}$$

one finds easily that, if p_0, q_0 and γp_0 are the coefficients of the term in s^n of $p_r(s)$, $k_1(1+\alpha s)\,q_r(s)$ and $r(s)$, then

$$\gamma^2 p_0=p_0+q_0\,. \tag{41}$$

Now define the polynomial $r_1(s)$, of degree $\leqq n-1$, by the relation

$$r(s)=\gamma p_r(s)+r_1(s)\,. \tag{42}$$

Substitution in (39) yields:

$$\begin{aligned}\frac{d}{dt}x_s'V_sx_s&=-r_1^2(D)z+p_r(D)z\left[k_1(1+\alpha D)\,q_r(D)z+p_r(D)z-\gamma^2 p_r(D)z-2\gamma r_1(D)z\right]\\&=-r_1^2(D)z-f(q_r(D)z)\cdot v(D)z\,,\end{aligned} \tag{43}$$

where degree $v(s)\leqq n-1$ because of (41).

Comparing (38) and (43) now yields the identity

$$x'(VA_r+A_r'V)x=-\left[r_1^2(D)z\right]_{x_s=M^{-1}x}$$

from which the argument proceeds as in Appendix 2. Finally one finds the conditions that:

1. $H(s)$ and $H(s+r)$ both have $m\geqq 1$ poles in the right half plane, while all other poles of $H(s+r)$ have negative real parts. (43)
2. The system $p_r(D)z=0$; $y=q_r(D)z$ is completely observable, which, in view of (40) and (42) is satisfied, since $p_r(s)$ and $(1+\alpha s)q_r(s)$ have no common factors**.

If we now generalize to the case $0\leqq k_0y^2\leqq yf(y)\leqq k_1y^2$; $k_0=df(y)/dy|_{y=0}$, then the system $(H(s);\,f(y))$ may be replaced by the equivalent system $(H_l(s);\,f(y)-k_0y)$, with

$$H_l(s)=\frac{H(s)}{1+k_0H(s)} \quad\text{and}\quad 0\leqq y(f(y)-k_0y)\leqq(k_1-k_0)y^2\,.$$

If $0\leqq k_1y^2\leqq yf(y)\leqq k_0y^2$; $k_0=df(y)/dy|_{y=0}$, then $(H(s);\,f(y))$ is replaced by $(-H_l(s);\,-f(y)+k_0y)$ where $0\leqq y(-f(y)+k_0y)\leqq(k_0-k_1)y^2$. If we apply the conditions (37), (38) and (43) to these equivalent systems, the instability criteria are obtained as stated in theorem 3.

* Note however that here we have: $K=0$, hence $g(y)=f(y)$; $A_l=A$; $H_l(s)=H(s)$.

** Assuming the parameters α and r are chosen such that $(1+\alpha s)$ is no factor of $p_r(s)$.

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